The zeta function of $H \times \mathbb{Z}^2$ counting all subrings

1 Presentation

$H \times \mathbb{Z}^2$ has presentation

$$\langle x, y, z, a, b \mid [x, y] = z \rangle .$$

$H \times \mathbb{Z}^2$ has nilpotency class 2.

2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$$\begin{aligned}\zeta_{H\times\mathbb{Z}^2,p}(s) = {} & \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(2s-4)\zeta_p(2s-5) \\ & \times \zeta_p(3s-5)^{-1}.\end{aligned}$$

$\zeta_{H\times\mathbb{Z}^2}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\zeta_{H\times\mathbb{Z}^2,p}(s)\big|_{p\to p^{-1}} = -p^{10-5s}\zeta_{H\times\mathbb{Z}^2,p}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta_{H\times\mathbb{Z}^2}(s)$ is 4, with a simple pole at $s = 4$.

5 Ghost zeta function

This zeta function is its own ghost.

6 Natural boundary

$\zeta_{H\times\mathbb{Z}^2}(s)$ has meromorphic continuation to the whole of $\mathbb{C}$.